Computational Procedures for Recent Analyses of Counterflow Heat Exchangers

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In a recent issue of the *Journal*, Nunge and Gill (*4*) discussed the results of an analysis of countercurrent laminar flow heat transfer in double-pipe heat exchangers. The analysis requires use of denumerably infinite sets of both positive and negative eigenvalues. Also, the generalized Fourier or expansion coefficients of the series solutions cannot be determined by simple integration, since they are expressible only in terms of the unknown fluid outlet temperature distributions.

Similar analytical research has been in progress at Argonne National Laboratory with emphasis on turbulent liquid metal flow. A detailed description of the progress made up to late 1964 is given in reference *6*; more recent results are discussed in reference *7*. These publications describe several differences between the Nunge–Gill and the Argonne treatment of the general counterflow problem. The most important of these differences concerns the procedures for determining the expansion coefficients; that is, the B_n^+ and B_n^- (*4*), the C_n (*6*), and the C_n and A_n (*7*). It is this writer's opinion that these procedures, and especially the particular procedure used by Nunge and Gill, require much more descriptive discussion than previously given (*4*) as well as in an earlier related paper (*2*). It is the absence of such discussion in a manuscript co-authored by Nunge, Gill, and this writer (*5*) that is the main reason why this joint effort remains unpublished.

It is the purpose of this note to indicate the kind of discussion believed to be required to justify those results of the analysis shown in reference *4* that depend upon the expansion coefficients (for example, Figures 5, 6, 7, 15, and 16 of reference *4*), and to suggest the possibility that these results may be inaccurate in some cases by comparing the procedure used by Nunge and Gill with an alternate procedure being used at Argonne.

THE NUNGE–GILL PROCEDURE

The procedure used by Nunge and Gill in reference *4*, as well as in reference *2*, requires the solution of an infinite set of simultaneous linear algebraic equations with the expansion coefficients as the unknowns; for example, Equation (38) in reference *4* or Equation (22) in reference *2* represents such sets with the B_q^+ and B_q^- as unknowns ($q, n = 0, 1, 2, 3, \ldots \infty$). A straightforward technique for investigating infinite sets of this type is to solve for k unknowns by using k equations, and to study the behavior of the unknowns as k is increased successively. If the "lower ordered" unknowns do not become independent of the number of equations used, then no unique solution exists. If the lower ordered unknowns become independent of the number of equations used, then a solution has been obtained or has "converged" for those particular unknowns. Convergence and accuracy, however, must be judged by purely inductive means by inspection of the results of the successive computations.

The statements made by Nunge and Gill (*2, 4*) that ". . . to find the expansion coefficients a set of simultaneous equations must be solved . . . each successive term added to the solution has an effect on the preceding expansion coefficients . . ." suggests either that their system of equations does not have a unique solution, or that an insufficient number of equations was used to obtain convergence in the sense described above.

Attempts at Argonne to use the Nunge–Gill procedure for plug flow problems have not been successful. A variety of formulations of the procedure always resulted in an "ill-conditioned" set of equations, and it is believed that as a result a sufficient number of equations could not be used to obtain convergence before round-off errors became prohibitive.

THE ARGONNE PROCEDURE

At Argonne a procedure, originated by Brown (*1*) and developed further by this writer, was found to be much more successful. Although this procedure also requires the solution of an infinite set of simultaneous equations, when properly formulated it results in a "well conditioned" set of equations. The basis for the Argonne procedure is described in reference *7*.

FORMULATION OF PROCEDURES

Application of separation of variables to the counterflow problem results in two expressions for the expansion coefficients of the infinite series solution: Equations (18) and (19) of reference *2*, Equation (36) and the one following of reference *3*, Equations (142) and (143) of reference *6*, and Equations (54) and (56) of reference *7*. When the two expressions are subtracted to eliminate the expansion coefficients, an infinite set of integral equations results relating the unknown outlet temperature distribution functions: Equation (37) of reference *3*, Equation (149) of reference *6*, and Equation (57) of reference *7*. The Nunge–Gill formulation is obtained by substitution of the appropriate series solutions for the unknown outlet temperature distributions in the set of integral equations.

The Argonne formulation is based on adding the two expressions for the expansion coefficients rather than subtracting them. The addition is made after transposing to the left-hand side the exponential term that appears in

TABLE 1. COMPARISON OF PROCEDURES FOR CASE OF $H = 0.5$, $K = 1.0$, $\epsilon^* = 0.8$

Even values of M: Argonne Procedure
Odd values of M: Nunge-Gill Procedure

M	Coeff. Det.*	ϵ	C_1	C_4	A_1	A_4
2	2.7 + 000	0.80254	–0.053890		0.576180	
3	*–5.8 – 002*	*0.71440*	*0.015700*		*0.693728*	
4	2.9 + 000	0.81848	–0.037706		0.565415	
5	*2.2 – 004*	*0.85882*	*–0.039451*		*0.518106*	
6	3.1 + 000	0.82191	–0.033509		0.562939	
7	*–2.9 – 008*	*0.81240*	*–0.029884*		*0.577085*	
8	3.2 + 000	0.82319	–0.031861	0.007801	0.561993	–0.023307
9	*1.2 – 013*	*0.82995*	*–0.027649*	*0.062377*	*0.554176*	*–0.164560*
10	3.3 + 000	0.82381	–0.031049	0.006827	0.561533	–0.022732
11	*–1.4 – 020*	*0.82292*	*–0.030933*	*–0.134433*	*0.563442*	*0.203826*
12	3.3 + 000	0.82415	–0.030589	0.006220	0.561274	–0.022371
13	*4.4 – 029*	*0.82590*	*–0.028330*	*0.242463*	*0.559475*	*–0.256252*
14	3.4 + 000	0.82437	–0.030302	0.005818	0.561113	–0.022131
15	*–3.8 – 039*	*0.82462*	*–0.030273*	*–0.365579*	*0.561181*	*0.175351*
28	3.7 + 000	0.82484	–0.029665	0.004857	0.560765	–0.021559
29	*1.7 – 154*	*0.82494*	*–0.032484*	*–7.794985*	*0.560755*	*0.190149*
30	3.7 + 000	0.82486	–0.029636	0.004809	0.560750	–0.021531
31	*–6.7 – 174*	*0.82499*	*–0.037626*	*–19.140512*	*0.560681*	*0.042927*
40	3.8 + 000	0.82493	–0.029545	0.004663	0.560702	–0.021449
41	*6.6.—275*	*0.82499*	*–0.032625*	*–8.971696*	*0.560679*	*0.039298*
42	3.9 + 000	0.82494	–0.029534	0.004645	0.560696	–0.021439
43	*–1.3 – 293*	*0.82502*	*–0.030160*	*–1.550187*	*0.560646*	*–0.007803*
48	3.9 + 000	0.82495	–0.029508	0.004603	0.560683	–0.021416
49	†	—	—	—	—	—
50	3.9 + 000	0.82496	–0.029501	0.004592	0.560679	–0.021410

* The notation 1.0-012 means 1.0×10^{-12}, etc.
† Absolute value less than 1.0-308, the lower limit for the CDC-3600.

one of the expressions: Equation (19) of reference 2, the expression following Equation (36) of reference 3, Equation (143) of reference 6, and Equation (56) of reference 7. The appropriate series solutions are then substituted for the unknown outlet temperature distributions. These manipulations, together with the overall heat balance relationship that exists for some of the constants involved [Equation (41) of reference 7], results in an infinite set of algebraic equations with the expansion coefficient corresponding to the zero eigenvalue eliminated.

Both formulations were made more convenient for computational purposes by avoiding the use of negative eigenvalues and by the use of normalized eigenfunctions. Negative eigenvalues were eliminated by simple redefinition of their corresponding expansion coefficients: Equations (59) to (61) of reference 7. Normalization was accomplished by dividing eigenfunctions by appropriate normalization factors (see footnote on p. 145 of reference 7).

COMPARISON OF PROCEDURES

Computations, pertaining to a heat exchanger consisting of adjacent parallel plane ducts with negligible thermal resistance through the common wall and plug flow models for both fluids, were performed to compare the two procedures for a large number of cases. The results of a much smaller number of comparative computations for a concentric tube exchanger with plug flow are briefly discussed in reference 7.

The sample results to be shown below are believed to be the near optimum obtainable with the Nunge-Gill procedure and required the use of double precision routines for the solution of the simultaneous equations, and normalization of the eigenfunctions (7). The alternate procedure also uses normalized eigenfunctions (7) but requires only single precision routines throughout.

Computations were performed on a CDC-3600 over a wide range of operating conditions as characterized by the dimensionless quantities, H, K, and ϵ^*. The parameters H and K are defined in references 6 and 7 and in earlier publications of this writer. The parameter H is the same in reference 4, and although K is defined somewhat differently in reference 4, it may be considered equivalent to the definition used here when applied to the parallel plane heat exchanger. The quantity ϵ^* represents a fictitious heat exchanger efficiency ($0 \leq \epsilon^* \leq 1$), and was used to specify the overall heat exchanger length by use of the customary heat exchanger design equation based on fully developed uniform heat flux heat transfer coefficients. As discussed (6, 7) the actual heat exchanger efficiency ϵ is related to the expansion coefficient C_o, corresponding to the zero eigenvalue. For $H < 1$, $\epsilon = 1 + (1 - H)C_o/H$. Because of the symmetrical geometry of the heat exchanger chosen, only values of H less than unity were considered. The two procedures were used to compute ϵ and the expansion coefficients A_n and C_n of reference 7 ($n = 1, 2, 3, \ldots$) as functions of the number of simultaneous equations M used to approximate the infinite set. For $H < 1$, the A_n are proportional to the B^-_{n-1} ($n = 1, 2, 3, \ldots$) of references 2 and 4, while the C_n are proportional to the B_n^+ ($n = 0, 1, 2, 3, \ldots$) of references 2 and 4. The proportionality results from the elimination of negative eigenvalues, and the use of normalized eigenfunctions.

With the Nunge–Gill procedure the use of M equations determines values for M of the expansion coefficients. The procedure was formulated so that an odd value of M corresponds to the determination of the expansion coefficient associated with the zero eigenvalue plus an equal number of expansion coefficients associated with negative and positive nonzero eigenvalues. Even values of M correspond to the determination of an even number of expansion coefficients associated with negative and positive eigenvalues, *including* the coefficient associated with the zero eigenvalue. When $H < 1$, the zero eigenvalue can

TABLE 2. COMPARISON OF PROCEDURES FOR CASE OF $H = 0.5$, $K = 0.1$, $\epsilon^* = 0.3$

Even values of M: Argonne Procedure
Odd values of M: Nunge-Gill Procedure

M	Coeff. Det.*	ϵ	C_1	C_2	A_1	A_2
2	9.5 + 000	0.41805	—0.174222		0.785483	
3	*5.0 — 002*	*0.77560*	*0.182611*		*0.396857*	
4	1.1 + 001	0.44321	—0.163791	0.091904	0.754193	—0.052804
5	*—1.1 — 003*	*0.36382*	*—0.233425*	*0.357757*	*0.839997*	*0.118906*
6	1.2 + 001	0.45213	—0.160709	0.089656	0.742510	—0.045845
7	*2.3 — 008*	*0.76315*	*0.089659*	*—1.151604*	*0.388904*	*0.727827*
8	1.2 + 001	0.45659	—0.159256	0.088676	0.736590	—0.041339
9	*—7.3 — 015*	*0.68087*	*—0.273266*	*1.322387*	*0.358596*	*—7.566796*
10	1.2 + 001	0.45920	—0.158429	0.088141	0.733111	—0.038179
11	*7.4 — 023*	*1.02566*	*0.031000*	*—0.154671*	*—0.025048*	*4.401746*
20	1.3 + 001	0.46395	—0.156960	0.087232	0.726732	—0.030868
21	*—2.7 — 099*	*1.00000*	*—0.000001*	*0.000003*	*0.000000*	*1.141015*
22	1.4 + 001	0.46433	—0.156845	0.087163	0.726223	—0.030175
23	*9.8 — 120*	*1.00000*	*0.000000*	*0.000000*	*0.000001*	*1.141194*
36	1.4 + 001	0.46565	—0.156445	0.086927	0.724439	—0.027593
37	*—1.5 — 268*	*1.00000*	*0.000000*	*0.000004*	*0.000000*	*1.144313*
38	1.4 + 001	0.46575	—0.156416	0.086910	0.724308	—0.027395
39	*2.0 — 291*	*1.00000*	*0.000002*	*—0.000014*	*—0.000002*	*1.145250*
46	1.5 + 001	0.46604	—0.156330	0.086860	0.723920	—0.026799
47	†	—	—	—	—	—
48	1.5 + 001	0.46609	—0.156314	0.086851	0.723847	—0.026686
49	†	—	—	—	—	—
50	1.5 + 001	0.46614	—0.156300	0.086842	0.723782	—0.026584

* The notation 1.0-012 means 1.0×10^{-12}, etc.
† Absolute value less than 1.0-308, the lower limit for the CDC-3600.

be considered to belong to the set of positive eigenvalues (7).

The Nunge–Gill procedure was found to be quite sensitive to the successive values of M chosen. For very small (or very large) values of the product KH the procedure failed completely, although odd values of M gave the best results. The procedure appeared to converge nonuniformly for the first few expansion coefficients when the product KH and ϵ^* (or ϵ) were not too far from unity. For values of KH very close to unity, even values of M gave the best results, otherwise odd values of M were best. In none of the cases explored would the Nunge–Gill procedure give values of the expansion coefficients A_n and C_n for $n > 3$.

The Argonne procedure does not include C_o as one of the unknowns in the infinite set of simultaneous equations. Instead C_o is computed from the expansion coefficients A_n and C_n ($n = 1, 2, 3, \ldots$) after they are determined. As a result when M simultaneous equations are solved, $M + 1$ values of the expansion coefficients are obtained. The simultaneous equations were chosen so that an equal number of expansion coefficients associated with nonzero negative and positive eigenvalues were included. Thus, M was always chosen as an even number.

The Argonne procedure gave uniformly converging values of the expansion coefficients for all of the cases explored and there appeared to be no limitation to the number of expansion coefficients that could be determined, provided a sufficiently large number of equations approximating the infinite set could be included. In many cases, however, convergence in the third to fifth decimal place was exceedingly slow, and accuracy would be difficult to judge without including an extremely large number of equations. As a result, improved computational procedures are being sought.

The results of the comparative computations are illustrated in Tables 1 and 2. The tables include the determinant of the coefficient matrix for both procedures. When the absolute value of this determinant is very small, round-off errors may become serious. It is believed that round-off errors are the main source of difficulty with the Nunge–Gill procedure. Table 1 shows results for a case with $KH = 0.5$ and $\epsilon^* = 0.8$ for which the Nunge–Gill procedure gives fairly good results for the first few expansion coefficients. Table 2 shows results for a case with $KH = 0.05$ and $\epsilon^* = 0.3$, for which the Nunge–Gill procedure fails completely. The reader is invited to study the results shown in these tables as related to the treatment of Equation (22) in reference 2 and Equation (38) in reference 4 as well as the tabulations of the expansion coefficients given in references 2 and 3.

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